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INTRODUCTION TO A NEW CLASS OF AGGREGATION OPERATORS ON MULTIPLE SETS

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Abstract. Multiple set theory has been introduced with the motive of getting a new mathematical tool that can easily handle the uncertainty of an object along with its multiplicity. This paper introduces a new class of Aggregation operators viz., Monotonic Anti Monotonic (MAM), Monotonic On Monotonic (MOM), and Monotone Identity Commutative Aggregation ($MICA$) on multiple sets. Also, some of their basic properties are discussed. Further, improved results are obtained by comparing t-norm, t-conorm, and uni-norm operators in multiple sets.

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1. Introduction

Multiple set theory introduced by Shijina and Sunil [7] is a unified mathematical structure to represent numerous uncertain features of objects simultaneously, in multiple ways. Specifically, each ambiguous aspect of an object is represented by a different fuzzy membership function and values are assigned to each membership function in line with the object's multiplicity. The ability to condense the whole data into a single matrix-like structure is the main advantage of multiple sets. Consequently, a matrix is assigned to each object by a multiple set where each row of the matrix denotes a unique fuzzy membership function according to each attribute of the item. The theoretical development of multiple sets along with a rudimentary introduction to aggregation operators, relations, similarity measures, and the topological structure of multiple sets were discussed in [7, 8, 9, 10, 11].

The theory of aggregation operations in fuzzy sets provides a useful account of how to combine several fuzzy sets in order to obtain a desirable fuzzy set. Several studies have been done around the theme of fuzzy aggregation operators [1, 2, 3, 4, 5]. Aggregation operators have been instrumental in the understanding of the problem of merging criteria functions to produce overall decision functions [1, 4]. The relationship between the various criteria is crucial in figuring out how such aggregation functions [6] should be structured. Major advancements in the study of aggregation operators establish the role of triangular norms in characterizing t-norm and t-conorm. Yager came up with the idea of generalizing connectives in fuzzy sets, introducing Monotonic Anti Monotonic (MAM), Monotonic On Monotonic (MOM),

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and Monotone Identity Commutative Aggregation ($MICA$) operators on fuzzy sets [16, 17, 18].

The aim of this study is to introduce MAM,MOM, and $MICA$ operators to the field of multiple sets. A conceptual theoretical framework has been developed with reference to Yager's work, defining MAM,MOM, and $MICA$ operators [12, 13, 14, 15] as bag mappings in multiple sets. Following this, a detailed comparison of MAM,MOM, and $MICA$ is made with the preset structures like t-norm, t-conorm, and matrix norm aggregation operators in multiple sets thereby bringing together the main findings. In the final section, a more simplified way for constructing a class of $MICA$ operators is depicted along with an example.

2. Preliminaries

Definition 2.1. [5, 12] *An aggregation operation on m fuzzy sets ($m \geq 2$) is a function $L : [0,1]^m \to [0,1]$ that satisfies the following axioms:*
i) $L(0,0,\ldots,0) = 0$ and $L(1,1,\ldots,1) = 1$.
ii)For any pair $(r_1, r_2, \ldots, r_m)$ and $(s_1, s_2, \ldots, s_m)$ of m-tuples such that $r_i, s_i \in [0,1]$, if $r_i \leq s_i$ for all $i \in N_m$ then $L(r_1, r_2, \ldots, r_m) \leq L(s_1, s_2, \ldots, s_m)$. i.e., L is monotonic increasing in all its arguments.
iii) L is continuous.
iv) $L(r_1, r_2, \ldots, r_m) = L(r_{p(1)}, r_{p(2)}, \ldots, r_{p(m)})$ for any permutation p on N_m.
v) L is idempotent function that is , $L(r, r, r, \ldots, r) = r$ for all $r \in [0,1]$.

Definition 2.2. [7] *Let Y be a non-empty crisp set called the universal set and $T_1, T_2, \ldots, T_n$ be n distinct fuzzy sets on Y corresponding to distinct attributes associated with each element in Y. For each $i \in N_n; T_i^1(y), T_i^2(y), \ldots, T_i^k(y)$ are membership values of the fuzzy set T for k identical copies of the element $y \in Y$ in decreasing order. Then a multiple set $\boldsymbol{T}$ of order (n,k) over Y is an object of the form $\{(y, \boldsymbol{T}(y)); y \in Y\}$ where for each $y \in Y$ its membership value is an $n \times k$ matrix in M(collection of all matrices) given by,*

$$\boldsymbol{T(y)} = \begin{bmatrix} T_1^1(y) & T_1^2(y) & \cdots & T_1^k(y) \\ T_2^1(y) & T_2^2(y) & \cdots & T_2^k(y) \\ \cdots & \cdots & \cdots & \cdots \\ T_n^1(y) & T_n^2(y) & \cdots & T_n^k(y) \end{bmatrix}$$

Definition 2.3. [7] *Let $\boldsymbol{G}, \boldsymbol{H} \in \boldsymbol{MS}_{(n,k)}(Y)$ where $\boldsymbol{MS}_{(n,k)}(Y)$ denotes the collection of all multiple sets of order (n,k) then,*
*i)**Subset:**$\boldsymbol{G} \subseteq \boldsymbol{H}$ if and only if $\boldsymbol{G}(y) \leq \boldsymbol{H}(y) \forall y \in Y$.*
*ii)**Union** : The union of $\boldsymbol{G}$ and $\boldsymbol{H}$ is a multiple set in $\boldsymbol{MS}_{(n,k)}(Y)$ denoted by $(\boldsymbol{G} \cup \boldsymbol{H})(y) = \boldsymbol{G}(y) \vee \boldsymbol{H}(y)$.*
*iii) **Complement**: Complement of $\boldsymbol{G} \in \boldsymbol{MS}_{(n,k)}(Y)$ denoted by $\bar{\mathbf{G}}$ whose membership matrix for each $y \in Y$ is an $n \times k$ matrix $\bar{\mathbf{G}}(y) = [\bar{\mathbf{G}}_{\mathbf{i}}^{\mathbf{j}}(\mathbf{y})]_{n \times k}$ where $\bar{\mathbf{G}}_i^j(y) = 1 - \mathbf{G}_i^{k-j+1} \forall i \in N_n$ and $j \in N_k$.*

Let $\mathbf{M^m}$ denote the Cartesian product $\mathbf{M} \times \mathbf{M} \times \ldots \times \mathbf{M}$($m$ times)

Definition 2.4. [8] *Let H_{ij} be fuzzy aggregation operators for every $i \in N_n$ and $j \in N_k$. Define a function $\mathbf{H} : \mathbf{M^m} \longrightarrow \mathbf{M}$ as follows,*
Matrices $N_1 = [(n_1)_{ij}]_{n\times k}, N_2 = [(n_2)_{ij}]_{n\times k}, \dots, N_m = [(n_m)_{ij}]_{n\times k}$ in $\boldsymbol{M}$ are mapped to a matrix $\boldsymbol{P}$=$[p_{ij}]_{n\times k}$ such that $p_{ij} = H_{ij}((n_1)_{ij}, (n_2)_{ij}, \dots, (n_m)_{ij})$ for every $i \in N_n$ and $j \in N_k$. Then $\boldsymbol{H}$ is called an aggregation operator induced by the fuzzy aggregation operator H_{ij} for every $i \in N_n$ and $j \in N_k$. It is represented as $\mathbf{H} = [H_{ij}]_{n\times k}$.

Definition 2.5. [14] *A mapping $E : I^n \to I$ is called an OWA (ordered weighted average) of dimension n if associated with E is a weighting vector $W = [W_1, W_2, \dots, W_n]^t$ such that*
i) $W_i \in (0, 1)$, ii) $\sum_i W_i = 1$ where $E(q_1, q_2, \dots, q_n) = W_1p_1 + W_2p_2 + \cdots + W_np_n$ and p_i is the i^{th} largest element in the collection $q_1, q_2, \dots, q_n$.

3. MAM, MOM and $MICA$ Operators.

Definition 3.1. *Consider X as the universal set and $A_1, A_2, \dots, A_m$ be the corresponding multiple sets of order (n, k) defined over X, then an m-bag in X is a collection of multiple sets where repetition is allowed. For example, let $\widehat{B} = \langle \mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}, \mathbf{A_2}\rangle$ then $\widehat{B}$ is a m-bag and $|\widehat{B}| = 4$.*

Definition 3.2. *Consider two m-bags $\widehat{A}$ and $\widehat{B}$, then their sum is defined as $\widehat{A} \oplus \widehat{B} = \{\mathbf{A_i} | \mathbf{A_i} \in \widehat{A}$ or $\mathbf{A_i} \in \widehat{B}\}$.*

Definition 3.3. *U^X denotes the collection of all m-bags defined over X. The m-bag operator $F : U^X \longrightarrow \boldsymbol{MS}_{(n,k)}(X)$ satisfying the following conditions,*
i) $F(\mathbf{A}) = \mathbf{A}$, ii) $F(\widehat{A} \oplus \widehat{B}) = F(\langle F(\widehat{A}), F(\widehat{B})\rangle)$ is called a value functional.

Definition 3.4. *Consider two m-bags $\widehat{A}$ and $\widehat{B}$, then $\widehat{A} \geq \widehat{B}$ if $\mathbf{A_i(x)} \geq \mathbf{B_i(x)}$ i.e. $A_i^j(x) \geq B_i^j(x) \forall x \in X$.*

Definition 3.5. *The m-bag mapping $H : U^X \longrightarrow MS_{(n,k)}(X)$ is an aggregation operator defined by $H\langle \mathbf{A_1}, \mathbf{A_2}, \dots, \mathbf{A_m}\rangle(\mathbf{x}) = \mathbf{P}(\mathbf{x})$ where $p_{ij} = H_{ij}[(a_1)_{ij}, (a_2)_{ij}, \dots, (a_m)_{ij}]$ and H_{ij} denotes the fuzzy aggregation operator. Hence the aggregation operator H is said to be induced by the fuzzy aggregation operator H_{ij}.*

Definition 3.6. *The m-bag mapping $H : U^Y \longrightarrow \boldsymbol{MS}_{(n,k)}(Y)$ is called a MAM operator if it satisfies the following conditions,*
i)If $\widehat{K} \geq \widehat{L}$ then $H(\widehat{K}) \geq H(\widehat{L})$.
ii) If $\widehat{D} = \widehat{K} \oplus \widehat{L}$ then $H(\widehat{K}) \geq H(\widehat{D})$.
iii) If $\widehat{D} = \widehat{K} \oplus \langle \mathbf{I}\rangle$ then $H(\widehat{D}) = H(\widehat{K})$ where $\mathbf{I}$ denote the multiple set whose membership matrix has value 1 in all entries $\forall x \in X$.
Monotonic Anti Monotonic (MAM) operator decreases the aggregated value with an increase in arguments.

Theorem 3.1. *MAM operator is a generalization of the "and" aggregation operator.*

Proof. Consider the mapping $H : U^Y \longrightarrow \mathbf{MS}_{(n,k)}(Y)$ defined by, $H(\widehat{R}) = min(\widehat{R})$, let $\widehat{R} = \langle \mathbf{R_1}, \mathbf{R_2}\rangle$ and $\widehat{S} = \langle \mathbf{S_1}, \mathbf{S_2}\rangle$.

(i)Take $\widehat{R} \geq \widehat{S}$ i.e., $\mathbf{R_i}(y) \geq \mathbf{S_i}(y); i = 1, 2$ *and* $y \in Y$.

$$\Longrightarrow min(\langle \mathbf{R_1}, \mathbf{R_2} \rangle) \geq min(\langle \mathbf{S_1}, \mathbf{S_2} \rangle).$$

Therefore $H(\widehat{R}) \geq H(\widehat{S})$.
That is, if $\widehat{R} \geq \widehat{S}$ then $H(\widehat{R}) \geq H(\widehat{S})$.

(ii) Let $\widehat{D} = \widehat{R} \oplus \widehat{S} = \langle \mathbf{R_1}, \mathbf{R_2}, \mathbf{S_1}, \mathbf{S_2} \rangle$.
Now, $H(\widehat{D}) = min\langle \mathbf{R_1}, \mathbf{R_2}, \mathbf{S_1}, \mathbf{S_2} \rangle \leq min\langle \mathbf{R_1}, \mathbf{R_2} \rangle = H(\widehat{R})$.
Hence, $H(\widehat{R}) \geq H(\widehat{D})$.

(iii) Let $\widehat{D} = \widehat{R} \oplus \langle \mathbf{I} \rangle = \langle \mathbf{R_1}, \mathbf{R_2}, \mathbf{I} \rangle$. That is, $H(\widehat{D}) = min\langle \mathbf{R_1}, \mathbf{R_2}, \mathbf{I} \rangle$. Since, $\mathbf{I(y)} \geq \mathbf{R_i}(y) \forall y \in Y, min(\langle \mathbf{R_1}, \mathbf{R_2}, \mathbf{I} \rangle) = min\langle \mathbf{R_1}, \mathbf{R_2} \rangle$ it follows that $H(\widehat{R}) = H(\widehat{D})$. □

Definition 3.7. *The m-bag mapping* $G : U^Y \longrightarrow \boldsymbol{MS}_{(n,k)}(Y)$ *is called as a* MOM *operator if it satisfies the following conditions,*
i)If $\widehat{J} \geq \widehat{K}$ *then* $G(\widehat{J}) \geq G(\widehat{K})$.
ii) If $\widehat{D} = \widehat{J} \bigoplus \widehat{K}$ *then* $G(\widehat{J}) \leq G(\widehat{K})$.
iii) If $\widehat{D} = \widehat{J} \bigoplus \langle \mathbf{O} \rangle$ *then* $G(\widehat{D}) = G(\widehat{J})$ *where* $\mathbf{O}$ *denote the multiple set whose membership matrix has 0 in all entries for all* $y \in Y$.
Monotonic On Monotonic (MOM) operator increases the aggregated value with an increase in arguments.

Theorem 3.2. *MOM operator is a generalization of the "or" aggregation operator.*

Proof. Consider the m-bag mapping $G : U^X \longrightarrow \mathbf{MS}_{(n,k)}(Y)$ defined by $H(\widehat{J}) = max(\widehat{J})$ where $\widehat{J} = \langle \mathbf{J_1}, \mathbf{J_2} \rangle$ and $\widehat{K} = \langle \mathbf{K_1}, \mathbf{K_2} \rangle$.

(i)Let $\widehat{J} \geq \widehat{K} \Longrightarrow max(\widehat{J}) \geq max(\widehat{K})$. That is, $G(\widehat{J}) \geq G(\widehat{K})$.

(ii) Let $\widehat{D} = \widehat{J} \bigoplus \widehat{K} = \langle \mathbf{J_1}, \mathbf{J_2}, \mathbf{K_1}, \mathbf{K_2} \rangle$.
Then $G(\widehat{D}) = max\langle \mathbf{J_1}, \mathbf{J_2}, \mathbf{K_1}, \mathbf{K_2} \rangle \geq max\langle \mathbf{J_1}, \mathbf{J_2} \rangle = G(\widehat{J})$.
Hence $G(\widehat{J}) \leq G(\widehat{D})$.

(iii)Let $\widehat{D} = \widehat{J} \bigoplus \langle \mathbf{O} \rangle = \langle \mathbf{J_1}, \mathbf{J_2}, \mathbf{O} \rangle$.
Since, $max(\langle \mathbf{J_1}, \mathbf{J_2} \rangle) = max(\langle \mathbf{J_1}, \mathbf{J_2}, \mathbf{O} \rangle)$ we get $G(\widehat{J}) = G(\widehat{D})$. □

A multiple t-norm and t-conorm are general versions of multiple $'and'$ and $'or'$ aggregation operators. Shijina et al. [8] introduced multiple t-norm and multiple t-conorm as binary operations on multiple sets. The above-defined MOM and MAM operators satisfy the conditions of multiple t-norm and multiple t-conorm. Indeed it can be considered as t-norm and t-conorm if the domain is restricted to the collection of bags consisting of two multiple sets only. Now we show the associativity and commutativity condition as follows:

Let $< \mathbf{P}, \mathbf{Q}, \mathbf{R} >$ be a bag consisting of multiple sets $\mathbf{P}, \mathbf{Q}$ and $\mathbf{R}$

Commutativity:

$$H(\mathbf{A},\mathbf{B}) = H(\mathbf{A}\oplus\mathbf{B}) = H(\mathbf{B}\oplus\mathbf{A}) = H(\mathbf{B},\mathbf{A}).$$

Associativity:

$$\begin{aligned} H(\mathbf{P},H(\mathbf{Q},\mathbf{R})) &= H(\mathbf{P},H(\mathbf{Q}\oplus\mathbf{R})) \\ &= H(H(\mathbf{P}),H(\mathbf{Q}\oplus\mathbf{R})) \\ &= H(H(\mathbf{P}\oplus\mathbf{Q}\oplus\mathbf{R})) \\ &= H(H(\mathbf{P}\oplus\mathbf{Q})\oplus H(\mathbf{R}))) \\ &= H(H((\mathbf{P}\oplus\mathbf{Q})\oplus\mathbf{R})) \\ &= H(H(\mathbf{P},\mathbf{Q}),\mathbf{R})) \\ &= H(H(\mathbf{P},\mathbf{Q}),\mathbf{R}). \end{aligned}$$

Definition 3.8. *The m-bag mapping $M : U^Y \longrightarrow \boldsymbol{MS}_{(n,k)}(Y)$ is called a $MICA$ operator if it has the following properties.*
i)If $\widehat{R} \geq \widehat{S}$ then $M(\widehat{R}) \geq M(\widehat{S})$.
ii) For every m-bag $\widehat{R}$ there exist an element $\langle\mathbf{U}\rangle \in \mathbf{U^Y}$ called the identity element of $\widehat{R}$ such that if, $\widehat{D} = \widehat{R}\oplus\langle\mathbf{U}\rangle$ then $M(\widehat{D}) = M(\widehat{R})$.
The Monotone Identity Commutative Aggregation ($MICA$) operator does not specify the identity element, it is mentioned as a general multiple set defined over X.

Theorem 3.3. *Matrix norm-aggregation operation defined in multiple set theory is a particular case of $MICA$ aggregation operator.*

Proof. Let $\mathbf{E},\mathbf{F},\mathbf{G}$ and $\mathbf{H}$ be multiple sets of same order defined over the universal set Y then,
Commutativity:

$$M(\mathbf{E},\mathbf{F}) = M(\mathbf{E}\oplus\mathbf{F}) = M(\mathbf{F}\oplus\mathbf{E}) = M(\mathbf{F},\mathbf{E}).$$

Monotonicity:

$$\begin{aligned} &For\ \ \mathbf{E}\geq\mathbf{F},\mathbf{G}\geq\mathbf{H} \text{ and } \mathbf{G}\geq\mathbf{E} \text{ the bag } \langle\mathbf{E},\mathbf{G}\rangle \geq \langle\mathbf{F},\mathbf{H}\rangle \\ &\qquad\qquad \Longrightarrow M\langle\mathbf{E},\mathbf{G}\rangle \geq M\langle\mathbf{F},\mathbf{H}\rangle. \end{aligned}$$

Associativity:

$$\begin{aligned} M(\mathbf{E},M(\mathbf{F},\mathbf{G})) &= M(\mathbf{E},M(\mathbf{F}\oplus\mathbf{G})) \\ &= M(M(\mathbf{E}),M(\mathbf{F}\oplus\mathbf{G})) \\ &= M(M(\mathbf{E}\oplus\mathbf{F}\oplus\mathbf{G})) \\ &= M(M(\mathbf{E}\oplus\mathbf{F})\oplus M(\mathbf{G}))) \\ &= M(M((\mathbf{E}\oplus\mathbf{F})\oplus\mathbf{G})) \\ &= M(M(\mathbf{E},\mathbf{F}),\mathbf{G})). \end{aligned}$$

Thus $M(\mathbf{E},M(\mathbf{F},\mathbf{G})) = M(M(\mathbf{E},\mathbf{F}),\mathbf{G})$.

The existence of identity directly follows from the definition of $MICA$-operator. □

Theorem 3.4. *For a $MICA$- operator M the identity element* $\mathbf{U}$ *is unique.*

Proof. Suppose $\mathbf{U_1}$ and $\mathbf{U_2}$ be two identity element then, $\widehat{P} = \langle \mathbf{U_1}, \mathbf{U_2} \rangle$, constitute a bag with 2 elements. Considering $\mathbf{U_1}$ as the identity element we get,

$$M(\widehat{P}) = \mathbf{U_2} \tag{1}$$

Considering $\mathbf{U_2}$ as the identity element we get,

$$M(\widehat{P}) = \mathbf{U_1} \tag{2}$$

From (1) and (2),

$$\mathbf{U_1} = \mathbf{U_2}.$$

□

Definition 3.9. *A $MICA$ operator M is said to have a fixed identity element if there exists a multiple set* $\mathbf{M}$ *such that* $\mathbf{U} = \mathbf{M} \ \forall \ \widehat{A}$.

Note 3.1. *MOM operator is a $MICA$ operator with fixed identity $\langle O \rangle$. MAM operator is a $MICA$ operator with fixed identity $\langle I \rangle$.*

Definition 3.10. *$MICA$ operator M is said to have a self identity if for every $\widehat{A}$ the identity* $\mathbf{U} = M(\widehat{A})$.

Example 3.1. *The average aggregation operator H is a self-identity aggregation operator.*
Consider the average aggregation operator H, let $\widehat{L}$ be an m-bag consisting of two multiple sets of order (2,2) defined over the universal set $Y = \{y\}$ then

$$H(\widehat{L}) = H\langle \mathbf{L_1}, \mathbf{L_2} \rangle$$

$$H\langle \mathbf{L_1} \rangle(y) = \begin{bmatrix} (y_1) & (y_2) \\ (y_3) & (y_4) \end{bmatrix}, \ H\langle \mathbf{L_2} \rangle(y) = \begin{bmatrix} (y_1)' & (y_2)' \\ (y_3)' & (y_4)' \end{bmatrix}$$

$$H\langle \mathbf{L_1}, \mathbf{L_2} \rangle(y) = \begin{bmatrix} (y_1 + (y_1)')/2 & (y_2 + (y_2)')/2 \\ (y_3 + (y_3)')/2 & (y_4 + (y_4)')/2 \end{bmatrix}$$

Let, $\widehat{D} = \widehat{L} \oplus H(\widehat{L}), H(\widehat{D}) = H(\widehat{L} \oplus H(\widehat{L}))$.

$$H\langle \mathbf{L_1}, \mathbf{L_2}, H(\widehat{L}) \rangle(y) = \begin{bmatrix} \frac{\{y_1 + (y_1)' + (y_1 + (y_1)')/2\}}{3} & \frac{\{y_2 + (y_2)' + (y_2 + (y_2)')/2\}}{3} \\ \frac{\{y_3 + (y_3)' + (y_3 + (y_3)')/2\}}{3} & \frac{\{y_4 + (y_4)' + (y_4 + (y_4)')/2\}}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \{3y_1 + 3(y_1)'\}/6 & \{3y_2 + 3(y_2)'\}/6 \\ \{3y_3 + 3(y_3)'\}/6 & \{3y_4 + 3(y_4)'\}/6 \end{bmatrix}$$

$$= \begin{bmatrix} \{y_1 + (y_1)'\}/2 & \{y_2 + (y_2)'\}/2 \\ \{y_3 + (y_3)'\}/2 & \{y_4 + (y_4)'\}/2 \end{bmatrix}$$

$$H\langle \mathbf{L_1}, \mathbf{L_2}, H(\widehat{L}) \rangle = H\langle \mathbf{L_1}, \mathbf{L_2} \rangle.$$

Theorem 3.5. *Let H be the $MICA$ aggregation operator with fixed identity* $\mathbf{I}$ *then for any m-bag $\widehat{S}$ and for any multiple set* $\mathbf{M}$ *we have,*

$$H(\widehat{S}) \geq H(\widehat{S} \oplus \langle \mathbf{M} \rangle).$$

Proof. Let $\widehat{D} = \widehat{S} \oplus \langle \mathbf{I} \rangle$ where $\mathbf{I}$ is the multiple set with each entry 1 in the membership matrix then,

$$H(\widehat{D}) = H(\widehat{S} \oplus \langle \mathbf{I} \rangle) = H(\widehat{S})$$

Now let, $\widehat{C} = \widehat{S} \oplus \langle \mathbf{M} \rangle$. That is $\widehat{C} \leq \widehat{D}$
From the monotonicity of H we have,

$$H(\widehat{C}) \leq H(\widehat{D})$$

$$\Longrightarrow H(\widehat{S}) \geq H(\widehat{S} \oplus \langle \mathbf{M} \rangle)$$

□

Theorem 3.6. *Let M be the MICA aggregation operator with fixed identity $\mathbf{O}$ then for any m-bag $\widehat{B}$ and for any multiple set $\mathbf{N}$ we have,*

$$M(\widehat{B}) \leq M(\widehat{B} \oplus \langle \mathbf{N} \rangle).$$

Proof. Let $\widehat{C} = \widehat{B} \oplus \langle \mathbf{O} \rangle$ where $\mathbf{O}$ is the multiple set with each entry 0 in the membership matrix then,

$$M(\widehat{C}) = M(\widehat{B} \oplus \langle \mathbf{O} \rangle) = M(\widehat{B}).$$

Now let, $\widehat{G} = \widehat{B} \oplus \langle \mathbf{N} \rangle$. That is $\widehat{G} \geq \widehat{C}$
From the monotonicity of M we have,

$$M(\widehat{G}) \geq M(\widehat{C})$$

$$\Longrightarrow M(\widehat{B}) \leq M(\widehat{B} \oplus \langle \mathbf{N} \rangle).$$

□

By Considering the idea of the conjugate of a multiple set, the dual aggregation operator of H can be defined as $\overline{H} : U^X \longrightarrow \mathbf{MS}_{(n,k)}(X)$ where $\overline{H}(\mathbf{P}) = (\mathbf{I} - H(\bar{\mathbf{P}}))^*$ and $\bar{\mathbf{P}} = \mathbf{I} - \mathbf{P}^*.\mathbf{P}^*$ is a matrix of order (n, k) with entries given by

$$\mathbf{P}^*(x) = \begin{bmatrix} (P)_1^k(x) & (P)_1^{k-1}(x) & \cdots & (P)_1^1(x) \\ (P)_2^k(x) & (P)_2^{k-1}(x) & \cdots & (P)_2^1(x) \\ \cdots & \cdots & \cdots & \cdots \\ (P)_n^k(x) & (P)_n^{k-1}(x) & \cdots & (P)_n^1(x) \end{bmatrix}$$

Theorem 3.7. *If H is a MICA operator with fixed identity $\mathbf{Q}$ then it's dual, $\bar{H}$ is also a MICA operator with fixed identity $\mathbf{I} - \mathbf{Q}^*$.*

Proof. Let $\widehat{A} = \langle \mathbf{Q_1} \rangle$ and $\widehat{B} = \langle \mathbf{Q_2} \rangle$ be two m-bags such that $\widehat{A} \geq \widehat{B}$.

$$\widehat{A} \geq \widehat{B} \Longrightarrow \mathbf{Q_1} \geq \mathbf{Q_2} \text{ hence, } \overline{\mathbf{Q_1}} \leq \overline{\mathbf{Q_2}}$$

$$\Longrightarrow H(\overline{\mathbf{Q_1}}) \leq H(\overline{\mathbf{Q_2}}) \tag{3}$$

Now, $\overline{H}(\mathbf{Q_1}) = (\mathbf{I} - H(\overline{\mathbf{Q_1}}))^*$and $\overline{H}(\mathbf{Q_2}) = (\mathbf{I} - H(\overline{\mathbf{Q_2}}))^*$.
From (3) we get $(\mathbf{I} - H(\overline{\mathbf{Q_1}}))^* \geq (\mathbf{I} - H(\overline{\mathbf{Q_2}}))^*$.

$$\Longrightarrow \overline{H}(\mathbf{Q_1}) \geq \overline{H}(\mathbf{Q_2}).$$

Let $\widehat{D} = \mathbf{Q_1} \oplus \langle \mathbf{I} - \mathbf{Q}^* \rangle$ then,

$$\overline{\widehat{D}} = \overline{\mathbf{Q_1}} \oplus \langle \overline{\mathbf{I} - \mathbf{Q}^*} \rangle$$
$$H(\overline{\widehat{D}}) = H(\overline{\mathbf{Q_1}} \oplus \langle \mathbf{Q} \rangle) = \mathbf{H}(\overline{\mathbf{Q_1}})$$
$$\overline{H}(\widehat{D}) = (\mathbf{I} - H(\overline{\widehat{D}}))^* = (\mathbf{I} - H(\overline{\mathbf{Q}}_1))^* = \overline{H}(\mathbf{Q_1}).$$

That is, $\overline{H}(\mathbf{Q_1}) = \overline{\mathbf{H}}(\mathbf{Q_1}) \oplus \langle \mathbf{I} - \mathbf{Q}^* \rangle$.

□

Theorem 3.8. *If R is a self identity $MICA$ operator then it's dual $\overline{R}$ is also a self identity $MICA$ operator.*

Proof.

$$\textit{Let } \widehat{B} = \mathbf{T} \oplus \langle \overline{R}(\mathbf{T}) \rangle \textit{ then } \overline{R}(\widehat{B}) = \overline{R}(\mathbf{T} \oplus \langle \overline{R}(\mathbf{T}) \rangle)$$
$$= (\mathbf{I} - R(\overline{\mathbf{T} \oplus \langle \bar{R}(\mathbf{T}) \rangle}))^*$$
$$= (\mathbf{I} - R(\overline{T} \oplus R(\overline{\mathbf{T}}))^*$$
$$= \mathbf{I} - R(\overline{\mathbf{T}})$$
$$= \overline{R}(\mathbf{T}).$$

Thus $\overline{R}$ is also a self-identity $MICA$ operator. □

3.1. **A group of ordered $MICA$ operators on multiple sets.** For any $\langle U \rangle \in U^X$ it's possible to define a $MICA$ operator M having $\langle U \rangle$ as the fixed identity. Let g be a non decreasing function defined on $\mathbf{MS}_{(n,k)}(X)$ then for any m-bag $\widehat{A} = \langle \mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_n} \rangle$ the operator $M(\widehat{A}) = g \sum_i |\overline{\mathbf{A_i} - \mathbf{U}}|$ where $|\overline{\mathbf{A_i} - \mathbf{U}}|$ is the dual of multiple set $\mathbf{A_i} - \mathbf{U}$ with entries in $[0,1]$. Thus, $M(\widehat{A})$ is a $MICA$ operator with fixed identity $\langle U \rangle$.
For any two bags $\widehat{P}$ and $\widehat{Q}$ with $\widehat{P} \geq \widehat{Q}$,

$$M(\widehat{P}) = g \sum_i |\overline{\mathbf{P_i} - \mathbf{U}}|.$$

$$M(\widehat{Q}) = g \sum_i |\overline{\mathbf{Q_i} - \mathbf{U}}|.$$

$$\text{Since} \quad \widehat{P} \geq \widehat{Q}, \sum_i |\overline{\mathbf{P_i} - \mathbf{U}}| \geq \sum_{\mathbf{i}} |\overline{\mathbf{Q_i} - \mathbf{U}}|.$$

$$\Longrightarrow M(\widehat{P}) \geq M(\widehat{Q}).$$

Let $\widehat{D} = \widehat{P} \oplus \langle \mathbf{U} \rangle$ then,

$$M(\widehat{D}) = g \sum_i |\overline{\mathbf{P_i} - \mathbf{U}}| + |\overline{\mathbf{U} - \mathbf{U}}| = M(\widehat{P}).$$

Thus M is a $MICA$ operator with fixed identity $\mathbf{U}$. Similarly, it is possible to construct an ordered weighted $MICA$ operator. Let $r : N \longrightarrow [0,1]$ be a non-decreasing function, the weighted matrix W of order (n,k) is given by

$$W = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1k} \\ w_{21} & w_{22} & \cdots & w_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ w_{n1} & w_{n2} & \cdots & w_{nk} \end{bmatrix}$$

where $w_{(ij)} = r(i+j+1) - r(i+j)$.
Now define the ordered weighted $MICA$ operator M' as,

$$M'(\widehat{A}) = g \sum_i |\overline{WP_i - WU}|.$$

Clearly, M' satisfies the properties of the $MICA$ operator with the fixed identity WU. Let,

$$\widehat{D} = \widehat{P} \oplus \langle WU \rangle.$$

$$M'(\widehat{D}) = g \sum_i |\overline{\mathbf{P}_i - WU}| + |\overline{WU - WU}| = M'(\widehat{P}).$$

Lets consider the non-decreasing functions g , h and i defined on $\mathbf{MS}_{(n,k)}(Z)$ where,

$$g(\mathbf{P}) = \begin{bmatrix} P_1^1/2 & P_1^2/2 & \cdots & P_1^k/2 \\ P_2^1/2 & P_2^2/2 & \cdots & P_2^k/2 \\ \cdots & \cdots & \cdots & \cdots \\ P_n^1/2 & P_n^2/2 & \cdots & P_n^k/2 \end{bmatrix}$$

$$h(\mathbf{P}) = \begin{bmatrix} (1+\ln(P_1^1))/2 & (1+\ln(P_1^2))/2 & \cdots & (1+\ln(P_1^k))/2 \\ (1+\ln(P_2^1))/2 & (1+\ln(P_2^2))/2 & \cdots & (1+\ln(P_2^k))/2 \\ \cdots & \cdots & \cdots & \cdots \\ (1+\ln(P_n^1))/2 & (1+\ln(P_n^2))/2 & \cdots & (1+\ln(P_n^k))/2 \end{bmatrix}$$

$$i(\mathbf{P}) = \begin{bmatrix} (1-P_1^1)^2 & 0 & \cdots & 0 \\ 1 & (1-P_2^2)^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & (1-P_n^k)^2 \end{bmatrix}$$

As mentioned above, it is possible to define $MICA$ aggregation operators α, β, and γ.

$$\alpha : U^Z \longrightarrow \mathbf{MS}_{(2,2)}(Z), \alpha(\widehat{P}) = g \sum_i |\overline{\mathbf{P_i} - \mathbf{U}}|.$$

$$\beta : U^Z \longrightarrow \mathbf{MS}_{(2,2)}(Z), \beta(\widehat{P}) = h \sum_i |\overline{\mathbf{P_i} - \mathbf{U}}|.$$

$$\gamma : U^Z \longrightarrow \mathbf{MS}_{(2,2)}(Z), \gamma(\widehat{P}) = i \sum_i |\overline{\mathbf{P_i} - \mathbf{U}}|.$$

Example 3.2. *Let $Z = \{z_1, z_2\}$ be the universal set.For the m-bag $\widehat{P} = \langle \mathbf{P_1}, \mathbf{P_2} \rangle$ of order $(2,2)$ defined over Z with,*

$$\mathbf{P_1}(z_1) = \begin{bmatrix} 0.3 & 0.2 \\ 0.2 & 0.1 \end{bmatrix}, \mathbf{P_1}(z_2) = \begin{bmatrix} 0.5 & 0.4 \\ 0.3 & 0.3 \end{bmatrix}$$

$$\mathbf{P_2}(z_1) = \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.2 \end{bmatrix}, \mathbf{P_2}(z_2) = \begin{bmatrix} 0.4 & 0.4 \\ 0.2 & 0.2 \end{bmatrix}$$

For,

$$U = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$|\overline{\mathbf{P_1} - \mathbf{U}}|(z_1) = \begin{bmatrix} 0.8 & 0.3 \\ 0.9 & 0.2 \end{bmatrix}$, $|\overline{\mathbf{P_1} - \mathbf{U}}|(z_2) = \begin{bmatrix} 0.6 & 0.5 \\ 0.7 & 0.3 \end{bmatrix}$

$|\overline{\mathbf{P_2} - \mathbf{U}}|(z_1) = \begin{bmatrix} 0.6 & 0.6 \\ 0.8 & 0.7 \end{bmatrix}$, $|\overline{\mathbf{P_2} - \mathbf{U}}|(z_2) = \begin{bmatrix} 0.6 & 0.6 \\ 0.8 & 0.2 \end{bmatrix}$

$$\alpha(\langle \mathbf{P_1}, \mathbf{P_2} \rangle)(z_1) = g \sum_i |\overline{\mathbf{P_i} - \mathbf{U}}|(z_1)$$

$$\alpha(\langle \mathbf{P_1}, \mathbf{P_2} \rangle)(z_1) = g(|\overline{\mathbf{P_1} - \mathbf{U}}|(z_1) + |\overline{\mathbf{P_2} - \mathbf{U}}|(z_1))$$

$$\alpha(\langle \mathbf{P_1}, \mathbf{P_2} \rangle)(z_1) = \begin{bmatrix} 0.7 & 0.45 \\ 0.85 & 0.45 \end{bmatrix}$$

$$\alpha(\langle \mathbf{P_1}, \mathbf{P_2} \rangle)(z_2) = g \sum_i |\overline{\mathbf{P_i} - \mathbf{U}}|(z_2)$$

$$\alpha(\langle \mathbf{P_1}, \mathbf{P_2} \rangle)(z_2) = g(|\overline{\mathbf{P_1} - \mathbf{U}}|(z_2) + |\overline{\mathbf{P_2} - \mathbf{U}}|(z_2))$$

$$\alpha(\langle \mathbf{P_1}, \mathbf{P_2} \rangle)(z_2) = \begin{bmatrix} 0.6 & 0.55 \\ 0.75 & 0.25 \end{bmatrix}$$

$$\beta(\langle \mathbf{P_1}, \mathbf{P_2} \rangle)(z_1) = h \sum_i |\overline{\mathbf{P_i} - \mathbf{U}}|(z_1)$$

$$\beta(\langle \mathbf{P_1}, \mathbf{P_2} \rangle)(z_1) = h(|\overline{\mathbf{P_1} - \mathbf{U}}|(z_1) + |\overline{\mathbf{P_2} - \mathbf{U}}|(z_1))$$

$$\beta(\langle \mathbf{P_1}, \mathbf{P_2} \rangle)(z_1) = \begin{bmatrix} 0.67 & 0.45 \\ 0.77 & 0.45 \end{bmatrix}$$

$$\beta(\langle \mathbf{P_1}, \mathbf{P_2} \rangle)(z_2) = h \sum_i |\overline{\mathbf{P_i} - \mathbf{U}}|(z_2)$$

$$\beta(\langle \mathbf{P_1}, \mathbf{P_2} \rangle)(z_2) = h(|\overline{\mathbf{P_1} - \mathbf{U}(\mathbf{z_2})|} + |\overline{\mathbf{P_2} - \mathbf{U}}|(z_2))$$

$$\beta(\langle \mathbf{P_1}, \mathbf{P_2} \rangle)(z_2) = \begin{bmatrix} 0.59 & 0.55 \\ 0.7 & 0.16 \end{bmatrix}$$

$$\gamma(\langle \mathbf{P_1}, \mathbf{P_2} \rangle)(z_1) = i \sum_i |\overline{\mathbf{P_i} - \mathbf{U}}|(z_1)$$

$$\gamma(\langle \mathbf{P_1}, \mathbf{P_2} \rangle)(z_1) = i(|\overline{\mathbf{P_1} - \mathbf{U}}|(z_1) + |\overline{\mathbf{P_2} - \mathbf{U}}|(z_1))$$

$$\gamma(\langle \mathbf{P_1}, \mathbf{P_2} \rangle)(z_1) = \begin{bmatrix} 0.016 & 0 \\ 1 & 0.01 \end{bmatrix}$$

$$\gamma(\langle \mathbf{P_1}, \mathbf{P_2} \rangle)(z_2) = i \sum_i |\overline{\mathbf{P_i} - \mathbf{U}}|(z_2)$$

$$\gamma(\langle \mathbf{P_1}, \mathbf{P_2} \rangle)(z_2) = i(|\overline{\mathbf{P_1} - \mathbf{U}}|(z_2) + |\overline{\mathbf{P_2} - \mathbf{U}}|(z_2)$$

$$\gamma(\langle \mathbf{P_1}, \mathbf{P_2} \rangle)(z_2) = \begin{bmatrix} 0.4 & 0 \\ 1 & 0.25 \end{bmatrix}$$

Similarly, ordered weighted $MICA$ operators can be constructed. Yager has done detailed work on this in fuzzy set theory which served as a major motivation to extend those theories to multiple set theory. The more generalized nature of MAM, MOM, and $MICA$ operators provides a simple approach to dealing with practical problems involving decision-making.

4. Conclusion

The fuzzy aggregation operator is a much-explored topic of research. In this paper, we wisely generalize the theory of fuzzy aggregation operators into multiple sets taking into account the necessity of dealing with a variety of qualities in the decision-making process and many other practical problems. Yager's quest to acquire more generic operators leads to the introduction of the MAM, MOM, and $MICA$ operators in fuzzy set theory. Here we introduce the notion of MAM, MOM, and $MICA$ operators on multiple sets, and their properties are investigated in detail. We have shown the equivalence of $MICA$ operators and matrix-norm aggregation operators defined in multiple set theory and the final results are quoted. At the end of the study, we presented a generalized method to derive ordered $MICA$ aggregation operators.

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